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Technical Notes

Temperature and Stress Fields for Short Pulse Heating of Solids

H. Al-Qahtani* and B. S. Yilbas†
King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

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Introduction

THE heat equation is hyperbolic in short durations and anomalies associated with the Fourier model are avoided through considering heat wave propagation. In addition, thermal expansion of the solid substrate results in thermal stress development in the heated region. To obtain the thermal stress developed in the solid, temperature field needs to be known. The closed-form solution of temperature field can be obtained through solving of the Cattaneo equation with the appropriate boundary conditions. The closed-form solution of temperature field enables to obtain the analytical solution of thermal stress distribution in the heated region. Moreover, the closed-form solution provides the relation between temperature and stress fields with the heating parameters in time and space domains. Consequently, investigation into analytical solution for the Cattaneo equation and thermal stress equation becomes essential. Considerable research studies were carried out to examine the hyperbolic heat conduction equation. The solution structure of a hyperbolic heat conduction equation was presented by Wang [1]. He showed that the contributions of the initial temperature distribution and source disturbance to the temperature field in the hyperbolic heat conduction were related to that of the initial rate of temperature change. A material-invariant version of the Maxwell–Cattaneo law was proposed by Christov [2]. He showed that the new formulation allowed the elimination of the heat flux, which in turn yielding a single equation for temperature field.

Thermal wave oscillations and thermal relaxation time determination in a hyperbolic heat transport model were examined by Ordonez–Miranda and Alvarado–Gil [3]. They showed that simple analytical expressions for the values of the maxima and minima of the oscillations as well as for the frequencies were present. The limitations of the application of the Cattaneo equation were investigated by Bright and Zhang [4]. They provided insight into the applicability of the Cattaneo equation from the perspectives of statistical approach, equilibrium and irreversible thermodynamics, and the experimental aspects. The thermal stress behavior due to short heating duration was investigated by Al-Huniti and Al-Nimr [5,6]. They used a numerical scheme to obtain temperature and corresponding thermal stress fields in the solid. Naji et al. [7] investigated thermal stress development in orthotropic cylinder due to short heating duration. The governing equations were solved numerically using implicit methods. They indicated that the transient thermal stresses generated inside the plate fluctuated between compressive and tensile quantities. Darabseh et al. [8] studied thermal stress development in unidirectional composites. The numerical method based on implicit finite difference scheme was used to calculate the temperature, displacement, and stress distributions within the cylinder. The short-time thermal effects on mechanical response of the solid surface due to the pulsed lasers were examined by Chen et al. [9]. The coupled transient thermoelasticity equations were solved numerically using the finite difference scheme. The boundary element analysis of coupled thermoelasticity with relaxation times in finite domain was introduced by Tehrani and Eslami [10]. They discretized the transformed equations using the boundary element method.

Yilbas and Pakdemir [11] obtained an analytical solution for the hyperbolic heat conduction equation using the perturbation method. However, temperature formulation was limited to certain ranges of space and time domain. Consequently, in the present study, analytical solutions of Cattaneo and stress equations are presented for a time exponentially varying short pulse. The closed-form solutions for temperature and stress distributions are obtained using the Laplace transformation method.

Mathematical Analysis

The Cattaneo model was developed on the notion of relaxing the heat flux, which is expressed as:

$$\varepsilon\frac{\partial q}{\partial t} = -q - K\nabla T \tag{1}$$

where ε is the relaxation time, K is the thermal conductivity, q is the heat flux vector, and T is temperature. The Cattaneo equation yields to the hyperbolic heat conduction equation for the temperature field, which is propagative with a speed $c = \sqrt{\frac{K}{\varepsilon\rho C_p}}$, where ρ is the density and C_p is the specific heat. Although the thermal properties of the substrate material are temperature dependent, the constant properties are assumed in the equation due to the simplicity. It should be noted that the use of the variable properties results in a nonlinear equation of energy transport, which may not be solved analytically to obtain the closed-form solution for temperature and stress distributions. Combining Eq. (1) with the energy equation $\rho C_p \frac{\partial T}{\partial t} + \frac{\partial q}{\partial x} = 0$, the resulting hyperbolic equation (telegraph) can be written in terms of temperature:

$$\alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} + \varepsilon\frac{\partial^2 T}{\partial t^2} \tag{2}$$

with initial and boundary conditions:

$$T = 0 \quad \text{at } t = 0 \tag{3}$$

$$q = f(t) \quad \text{at } x = 0 \tag{4}$$

Taking the Laplace transformation of the governing Eq. (2), initial and boundary conditions [Eqs. (3) and (4)] result in:

$$\alpha\frac{\partial^2 \hat{T}}{\partial x^2} - s^2\varepsilon\hat{T} - s\hat{T} = 0 \tag{5}$$

$$\hat{q} = -K\frac{\partial \hat{T}}{\partial x} = F(s) \quad \text{at } x = 0 \tag{6}$$

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*Assistant Professor, Mechanical Engineering Department, Box 1913.
†Professor, Mechanical Engineering Department, Box 1913; bsyilbas@kfupm.edu.sa.

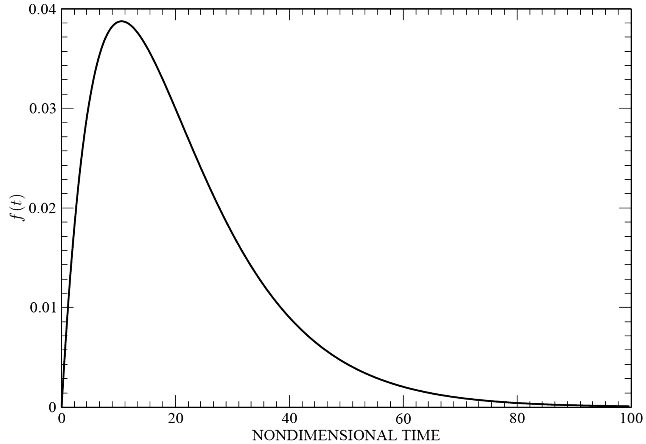

Fig. 1 Exponential pulse.

Pulse

$$f(t) = e^{-\beta_1 t} - e^{-\beta_2 t} \tag{7}$$

Because the model is linear, one can use the superposition principle to compute the temperature distribution due to the two terms of the pulse function separately and add them to get the final solution. Figure 1 shows the pulse used in the analysis. In the sequel, one term namely, $e^{-\beta t}$ is considered as a pulse source and then the total solution is simply

$$T = T|_{e^{-\beta_1 t}} - T|_{e^{-\beta_2 t}} \tag{8}$$

Temperature Distribution

The solution of the governing equation in the s-domain can be represented as:

$$\hat{T}(x,s) = C_1 e^{-\frac{x}{\sqrt{\alpha}}\sqrt{\varepsilon s^2 + s}} + C_2 e^{\frac{x}{\sqrt{\alpha}}\sqrt{\varepsilon s^2 + s}} \tag{9}$$

For the solution to be bounded C_2 must vanish.

By applying the transformed boundary condition, the constant C_1 can be determined and the solution in the transformed domain can be finally written as

$$\hat{T}(x,s) = \frac{\sqrt{\varepsilon\alpha}}{K(s+\beta)\sqrt{s(s+1/\varepsilon)}} e^{-\frac{x}{\sqrt{\alpha/\varepsilon}}\sqrt{s(s+1/\varepsilon)}} \tag{10}$$

The temperature distribution can be written as a multiplication of two functions as

$$\hat{T}(x,s) = \hat{T}_1(x,s)\hat{T}_2(x,s) \tag{11}$$

The inverse of the two terms can individually be written as

$$\begin{aligned}\mathcal{L}^{-1}\hat{T}_1(x,s) &= \mathcal{L}^{-1}\left(\frac{1}{\sqrt{s(s+1/\varepsilon)}} e^{-\frac{x}{\sqrt{\alpha/\varepsilon}}\sqrt{s(s+1/\varepsilon)}}\right) \\ &= e^{-\frac{t}{2\varepsilon}} I_o\left(\frac{1}{2\varepsilon}\sqrt{t^2 - \frac{x^2}{\alpha/\varepsilon}}\right) U\left(t - \frac{x}{\sqrt{\alpha/\varepsilon}}\right)\end{aligned} \tag{12}$$

and

$$\mathcal{L}^{-1}(\hat{T}_2(x,s)) = \mathcal{L}^{-1}\frac{\sqrt{\alpha\varepsilon}}{K(s+\beta)} = \frac{\sqrt{\alpha\varepsilon}}{K} e^{-\beta t} \tag{13}$$

where I_o is the modified Bessel function of the first kind, U is the unit step function.

The overall temperature distribution is obtained using the convolution

$$T(x,t) = \int_0^t \hat{T}_1(\tau)\hat{T}_2(t-\tau)\,\mathrm{d}\tau \tag{14}$$

Thermal Stresses

The thermoelasticity is a temperature rate dependent by including temperature rate among the constitutive variables [12]. However, most of the materials, the strain rate is of the same order of the temperature rate; in which case, the thermo-mechanical coupling becomes small for the small displacements and it can be neglected [8–12]. This simplifies the physical problem to yield the closed-form solution for the thermal stress field in the solid substrate. Consequently, after assuming the homogenous isotropic structure, the temperature distribution of the semi-infinite media in the s-domain can be represented in a nondimensional form:

$$\hat{T}(x,s) = \frac{(s+2)e^{-\sqrt{s^2+2s}x}}{(s+\beta)\sqrt{s^2+2s}} \tag{15}$$

The corresponding nondimensional stress field equation in the s-domain is given by

$$\hat{\sigma}(x) - \frac{s^2\hat{\sigma}(x)}{c_1^2} = \frac{c_2 s^2(s+2)}{(s+b)\sqrt{s^2+2s}} e^{-\sqrt{s(s+2)x}} \tag{16}$$

and c_1 and c_2 are the nondimensional wave speed and the nondimensional thermal modulus, respectively.

$$\hat{\sigma} = B_1 e^{\frac{sx}{c_1}} + B_2 e^{-\frac{sx}{c_1}} + \frac{c_2 s^2(s+2)}{(s+b)\sqrt{s^2+2s}} \frac{c_1^2 e^{-\sqrt{s(s+2)x}}}{s(c_1^2(s+2)-s)} \tag{17}$$

For boundedness of the solution $B_1 = 0$, and the condition of free stress (i.e., $\sigma = 0$) on the surface is imposed as this stage to calculate the other coefficient B_2 as

$$B_2 = -\frac{c_1^2}{s(c_1^2 s + 2c_1^2 - s)} \frac{c_2 s^2(s+2)}{(s+b)\sqrt{s^2+2s}} \tag{18}$$

Therefore

$$\begin{aligned}\hat{\sigma}(x,s) = &-\frac{c_1^2 c_2 s(s+2)e^{-\frac{sx}{c_1}}}{\sqrt{s^2+2s}(c_1^2 s + 2c_1^2 - s)(s+\beta)} \\ &+ \frac{c_1^2 c_2 s(s+2)e^{-\sqrt{s(s+2)s}}}{\sqrt{s^2+2s}(c_1^2(s+2)-s)(s+\beta)}\end{aligned} \tag{19}$$

where $\hat{\sigma}(x,s) = \hat{\sigma}_h(s,x) + \hat{\sigma}_p(x,s)$. The stress component $\hat{\sigma}_h$ can be written as a multiplication of two subcomponents as $\hat{\sigma}_h(x,s) = \hat{\sigma}_h^{(1)}(s,x)\hat{\sigma}_h^{(2)}(x,s)$. The inverse Laplace transform of the two subcomponents is carried out as the following:

$$\begin{aligned}\sigma_h^{(1)}(x,t) = \mathcal{L}^{-1}(\hat{\sigma}_h^{(1)}) = \frac{c_1^2 c_2}{c_1^2(\beta-2)-\beta}\Bigg(\beta(\beta-2)e^{\frac{x\beta}{c_1}-t\beta} \\ - \frac{4c_1^2}{(c_1^2-1)^2} e^{\frac{2c_1(x-c_1 t)}{c_1^2-1}}\Bigg) U\left[t - \frac{x}{c_1}\right] - \frac{c_1^2 c_2}{c_1^2-1}\delta\left[t - \frac{x}{c_1}\right]\end{aligned} \tag{20}$$

$$\sigma_h^{(2)}(x,t) = \mathcal{L}^{-1}(\hat{\sigma}_h^{(2)}) = e^{-t}J_0[\mathrm{it}] \tag{21}$$

One can get $\hat{\sigma}_h(x,t)$ by employing the convolution theorem:

$$\sigma_h(x,t) = \int_0^t \hat{\sigma}_h^{(1)}(\tau)\hat{\sigma}_h^{(2)}(t-\tau)\,\mathrm{d}\tau \tag{22}$$

Similarly, the second component of the stresses $\hat{\sigma}_p$ can be written can be written as a multiplication of two subcomponents:

$$\hat{\sigma}_p(x,s) = \hat{\sigma}_p^{(1)}(x,s)\hat{\sigma}_p^{(2)}(x,s) \tag{23}$$

The inverse Laplace transform is carried out for both subcomponents

$$\sigma_p^{(1)}(x,t) = \mathcal{L}^{-1}(\hat{\sigma}_p^{(1)}) = c_1^2 c_2 \left(\frac{1}{c_1^2 - 1} \delta[t] + \frac{(2\beta - \beta^2)}{c_1^2\beta - 2c_1^2 - \beta} e^{-\beta t} + \frac{4c_1^2}{(c_1^2 - 1)^2 (c_1^2\beta - 2c_1^2 - \beta)} e^{-\frac{2c_1^2}{c_1^2 - 1} t} \right) \tag{24}$$

$$\sigma_p^{(2)}(x,t) = \mathcal{L}^{-1}(\hat{\sigma}_p^{(2)}) = e^{-t} I_0(\sqrt{t^2 - x^2}) U[t - x] \tag{25}$$

One can get $\hat{\sigma}_p(x,t)$ by employing the convolution theorem

$$\sigma_p(x,t) = \int_0^t \hat{\sigma}_p^{(1)}(\tau) \hat{\sigma}_p^{(2)}(t - \tau)\, \mathrm{d}\tau \tag{26}$$

All components of the stresses have been transformed to $(x,\ t)$ domain, and we are ready to get the stresses as

$$\sigma(x,t) = \sigma_h(x,t) + \sigma_p(x,t) \tag{27}$$

A computer code is developed to compute temperature and stress distributions from Eqs. (14) and (27), respectively.

Results and Discussion

The closed-form solutions of Cattaneo and thermal stress equations are presented for time exponentially varying short pulse. The Laplace transformation method is used in the mathematical analysis.

Figure 2 shows temporal variation of dimensionless temperature at different locations inside the substrate material. The peak temperature attains the maximum at the surface and as the locations move at some distance below the surface, the peak temperature reduces. Moreover, the location of the peak temperature below the surface changes in time due to the finite speed of temperature in the substrate material. In general, the material response to the heating pulse at the surface is slow, because the maximum peak pulse intensity occurs at dimensionless time $t = 10$, while the peak temperature occurs at around dimensionless time $t \approx 25$ at the surface. This is associated with the pulse profile and the energy transfer taking place in the surface region during the heating cycle of the short pulse. In this case, the energy deposited to the substrate material during the early heating duration is not sufficient to rise temperature at the same rate as the rate of pulse rise. As the heating period progresses, the rate of temperature rise becomes high, particularly at the surface. The internal energy gain in the surface vicinity from the heating pulse is responsible for the high rate of temperature rise at the surface. However, as the heating period progresses further, temperature decays from its peak value. The rate of temperature decay is slower than that corresponding to the heating pulse. This is particularly true at some depth below the surface. In this case, the internal energy gain in the surface vicinity results in the attainment of high temperature and diffusional energy transfer from the surface region to the solid bulk takes place in a finite speed, which is less than the decay rate of the pulse intensity. As the pulse intensity decays beyond 5% of its peak intensity, temperature decay becomes gradual. Consequently, the temporal gradient of temperature becomes small, the Cattaneo equation tends to reduce the Fourier equation. Therefore, diffusional energy transport with infinite speed governs the energy transfer in the substrate material. This is more pronounced in the region below the surface.

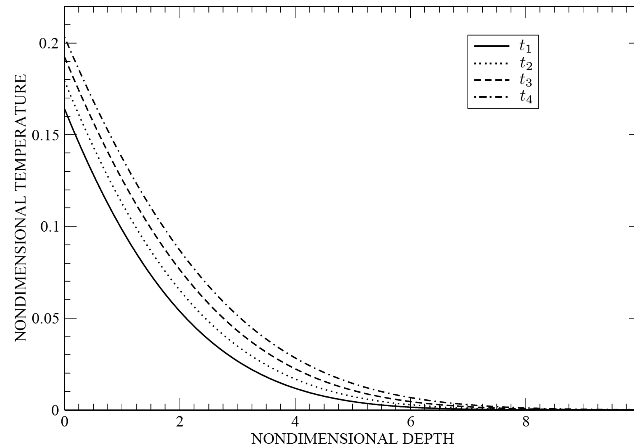

Fig. 3 Temperature distribution vs depth at different times.

Figure 3 shows dimensionless temperature distribution inside the substrate material for different heating periods. Temperature decays sharply in the substrate material. As the distance below the surface of the substrate material increases beyond the dimensionless distance $x = 4$, temperature decay becomes gradual. The sharp decay of temperature in the surface region is associated with the energy transfer from the surface vicinity to the solid bulk. In this case, the high temperature gradient generated in the surface vicinity is responsible for the heat diffusion from the surface vicinity to the solid bulk. Because the heating duration is short, the amount of heat transferred through conduction from the surface region to the solid bulk is not considerable. This, in turn, suppresses the energy transfer from the surface region to the solid bulk. Consequently, temperature gradient remains high in the region next to the surface vicinity. This situation is true for all the heating periods shown in the figure. However, the gradual decay in temperature at some depth below the surface is associated with the diffusional energy transfer in this region. The amount of heat transfer in this region is small during the short heating duration. This, in turn, lowers the temperature gradient and the diffusional energy transfer in this region.

Figure 4 shows temporal behavior of dimensionless thermal stress at different locations inside the substrate material. The compressive stress waves are formed first in the surface vicinity and they moved into the substrate material as the heating period progresses. The compression due to initial heating results in the formation of the compressive waves. However, the expansion of the initially

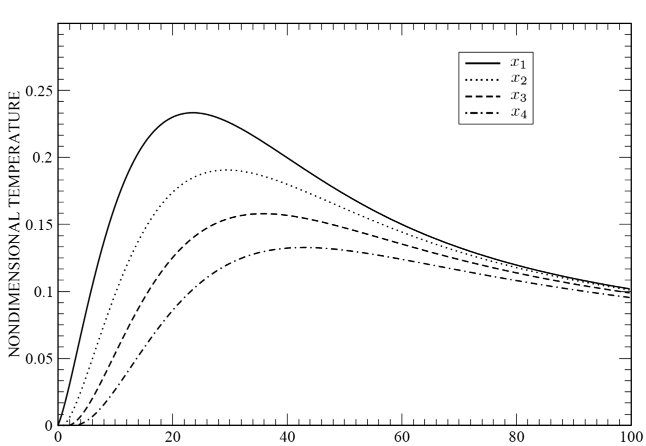

Fig. 2 Temperature distribution vs time at different depths.

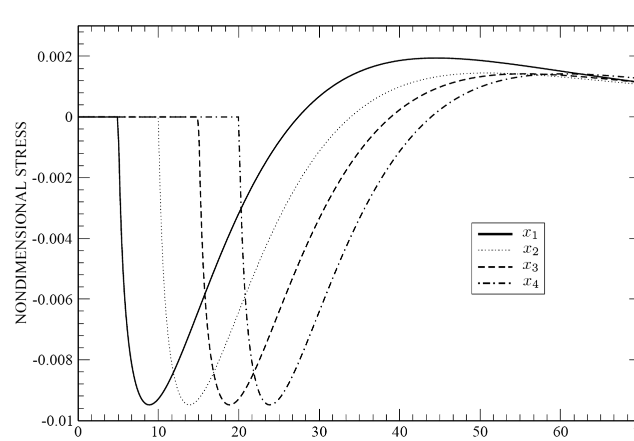

Fig. 4 Nondimensional stresses vs time at different locations.

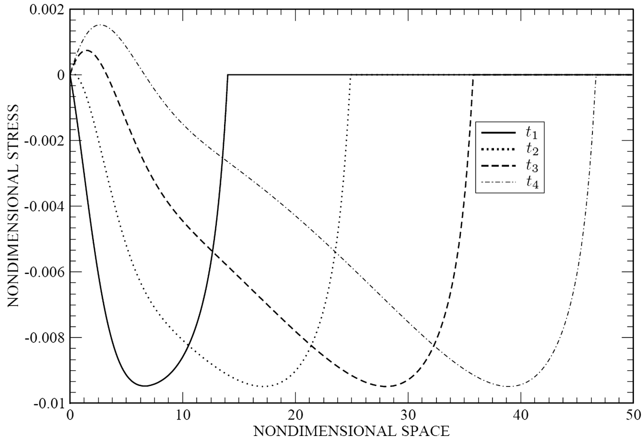

Fig. 5 Nondimensional stresses vs space at different times.

compressed surface results in tensile wave formation as the time progresses. This appears as positive wave amplitude in the tail of the stress wave. This situation is true for all the locations below the surface. The magnitude of the compressive wave is larger than the amplitude of the tensile part of the wave. This indicates the initially formed wave propagates in compression form in to the substrate material. Moreover, the rate of rise of the amplitude of the compressive wave is higher than that of the decay rate of the wave. This behavior is similar to temperature response of the heated substrate material. Consequently, rapid heating and gradual cooling of the substrate material in the surface region generates a compressive wave in a similar fashion, provided that the rates of temperature rise and decay are not the same as the rates of rise and decay of the thermal stresses.

Figure 5 shows dimensionless stress distribution inside the substrate material for different heating periods. It is evident that in the early heating periods, a compressive stress is generated in the surface region and the maximum compressive stress amplitude occurs at some depth below the surface. The location of the maximum stress amplitude moves away from the surface with progressing time. Moreover, the behavior of the compressive stress in the early heating period is slightly different; in which case, the compressive stress decays sharply resulting in high stress gradient in the surface region. As the heating period progresses, the stress field below the surface is modified due to the change of the temperature gradient below the surface. Consequently, gradually decaying of the thermal stress takes place in the region next to the surface vicinity. Moreover, as the heating period progresses further, thermal stress generated in the surface region becomes tensile. This is associated with the thermal expansion of the surface after the initial compression. However, the amplitude of the tensile wave is considerably smaller than that of the compressive wave. In addition, the tensile wave is only limited to the surface vicinity and as the depth below the surface increases, the thermal stress becomes compressive. This indicates that small expansion of the surface takes place during the heating period while generating the tensile stress in the surface region. However, the material below the surface remains in compression causing a compressive thermal stress development in this region.

Conclusions

The analytical solution for Cottaneo equation is presented for the time varying exponential short pulse heating of the solid substrate. The closed-form solutions for temperature and stress fields are obtained using the Laplace transformation method. It is found that temperature rise in the early heating period is gradual. As the heating period progresses, temperature rises rapidly reaching its maximum, which is more pronounced at the surface. The time occurrence of the peak temperature changes as the distance below the surface increases. In addition, the value of the peak temperature reduces at different locations below the surface. This is associated with: 1) the heating pulse intensity distribution, which reduces with progressing time, and 2) the energy transfer from the surface region to the solid bulk, which is not considerable due to the short heating duration. Temperature decay rate becomes small when the pulse intensity reduces 5% of its peak value. In this case, temporal gradient of temperature becomes small and the wave nature of the heating replaces with the diffusional heating. Temperature decay is sharp inside the substrate material during the heating cycle. This results in large temperature gradients and stress field below the surface. Temporal behavior of thermal stress reveals that the compressive stress waves are formed due to initial contraction of the surface during the early heating period. The compression wave reaches its peak value rapidly and decays gradually similar to the pulse intensity distribution. The stress wave generated propagates at a constant speed inside the substrate material. The thermal expansion of the surface during the late heating period results in the tensile wave formation in the surface region. This appears as a tensile tail in the compressive wave generated earlier. The magnitude of the tensile wave is significantly lower than that of the compressive wave.

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